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Coupled Motion in Alternating Phase Focused Linacs

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Synchrobetatron resonances in alternating phase focused (APF) linacs are studied with the equations of motion obtained by the stepping field approximation method. It is shown that symmetric phase alternation patterns cause the lowest order resonance which is the most undesirable. The width of the unstable region corresponding to this resonance is evaluated approximately.

KEY WORDS: APF linac/Synchrobetatron coupling/Resonance/

1. INTRODUCTION

An APF linac is a modified version of usual drift tube linacs (DTL) and can achieve three-dimensional focusing without the installation of quadrupole lenses into the drift tubes. This fact means that it is possible to operate an APF linac at higher frequency or in lower energy region compared with normal type DTLs. The APF acceleration method was discovered about forty years ago[1] and has been developed mainly by Russian researchers. Although the initial investigation of an APF linac concluded that the APF structure may not be practical because of its small longitudinal acceptance, further studies showed the possible APF designs called asymmetric APF (AAPF) and modified APF (MAPF)[2] which can realize larger acceptance comparable to normal DTLs. Various theoretical approachs to understand the beam dynamics of AAPF and MAPF have been proposed and APF linacs are now thought to be useful in the energy region below around several MeV/u.[3)~6)]

In APF linacs, intergap rf fields are used to achieve not only acceleration but also beam focusing, so, in the design stage, the consideration of the synchrobetatron couping becomes more important than that in normal DTLs[7]. The smoothed transverse phase advance σ_t^{DTL} in a normal DTL is separated into the two parts, *i.e.* $\sigma_t^{\mathrm{DTL}} = \sigma_t^{\mathrm{QM}} - \sigma_t^{\mathrm{RFD}}$. σ_t^{QM} is the phase advance of quadrupole focusing channel while σ_t^{RFD} comes from intergap rf fields. Because the strongest resonance occurs when $4(\sigma_t^{\mathrm{DTL}})^2 \sim (\sigma_1^{\mathrm{DTL}})^2 = 2(\sigma_t^{\mathrm{RFD}})^2$ and, in recent DTL designs, σ_t^{QM} makes a dominant contribution to σ_t^{DTL}, it is usually unnecessary to worry about the synchrobetatron couping effects. However, there is no σ_t^{QM} part in the APF phase advance and we must pay much attention to the coupled motion in APF structures. In the present paper, we investigate the synchrobetatron coupling effects in APF linacs, starting with the same type of equations presented in Ref. 3. The width of the unstable region around the lowest order resonance is also described.

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2. COUPLED EQUATIONS OF MOTION

The equations of motion of an APF linac with axisymmetric accelerating field can be represented as follows (see APPENDIX A and B);

$$\begin{cases} \dfrac{d^2x}{d\tau^2}+\dfrac{2A}{\kappa}\mathrm{I}_1(\kappa x)\sum_{k=1}^{N} V_k T_k \Lambda_k(\tau)\sin(\Delta\phi+\phi_k{}^s)=0 & (1.a) \\ \dfrac{d^2(\Delta\phi)}{d\tau^2}+2A\sum_{k=1}^{N} V_k T_k \Lambda_k(\tau)\ [I_0(\kappa x)\cos(\Delta\phi+\phi_k{}^s)-\cos\phi_k{}^s]=0 & (1.b) \end{cases}$$

where

$$A=\frac{\pi eqL_N}{m_0c^2\beta^3\gamma^3\lambda}.$$

Eqs.(1) show the strong coupling between transverse and longitudinal particle motion. If we keep only the linear terms in x and $\Delta\phi$, these equations agree with those presented in Ref.3. Expanding the Bessel functions, sine and cosine functions in eqs.(1) into power series, we obtain

$$\frac{d^2x}{d\tau^2}+K_s(\tau)x=8P_2(\tau)\sum_{n=2}^{\infty} nh_n\frac{(\kappa x)^{2n-1}}{\kappa}-4\sum_{n=1}^{\infty}P_n(\tau)(\Delta\phi)^n\sum_{m=1}^{\infty} mh_m\frac{(\kappa x)^{2m-1}}{\kappa} \qquad (2.\ a)$$

and

$$\frac{d^2(\Delta\phi)}{d\tau^2}-2K_s(\tau)(\Delta\phi)=-2\sum_{n=3}^{\infty} nP_n(\tau)(\Delta\phi)^{n-1} -2\sum_{n=1}^{\infty} nP_n(\tau)(\Delta\phi)^{n-1}\sum_{m=1}^{\infty} h_m(\kappa x)^{2m} \qquad (2.\ b)$$

where

$$h_n=\frac{1}{n!\Gamma(n+1)}\left(\frac{1}{2}\right)^{2n}, \quad P_n(\tau)=\begin{cases} \dfrac{(-1)^{n/2}}{n!}\cdot \mathrm{K}_s(\tau) & \text{for } n=even \\ \dfrac{(-1)^{(n-1)/2}}{n!}\cdot K_c(\tau) & for\ n=odd, \end{cases}$$

and

$$\begin{cases} K_s(\tau)=A\sum_{k=1}^{N} V_k T_k \Lambda_k(\tau)\sin\phi_k{}^s \\ K_c(\tau)=A\sum_{k=1}^{N} V_k T_k \Lambda_k(\tau)\cos\phi_k{}^s. \end{cases}$$

$\mathrm{K}_s(\tau)$ and $\mathrm{K}_c(\tau)$ are rewritten as

$$\begin{cases} K_s(\tau)=B+\sum_{n=1}^{\infty} C_n \sin(2n\pi\tau+\theta_n) \\ K_c(\tau)=B'+\sum_{n=1}^{\infty} C_n' \sin(2n\pi\tau+\theta'_n) \end{cases}$$

where θ_n and θ_n' are constant,

$$B=\sum_{k=1}^{N}\Delta_k=\sum_{k=1}^{N}A\,V_kT_k\sin\phi_k^s,$$

$$C_n=2\left\{\left[\sum_{k=1}^{N}\Delta_kS_{nk}\sin(2n\pi\tau_k)\right]^2+\left[\sum_{k=1}^{N}\Delta_kS_{nk}\cos(2n\pi\tau_k)\right]^2\right\}^{1/2},$$

and B′ and C_n' are obtained by replacing Δ_k in the above formulae by $\Delta_k'=AV_kT_k\cos\phi_k^s$. In the following calculations, we consider only the second order terms in x and $\Delta\phi$ in the r.h.s. of eqs.(2), *i.e.*

$$\begin{cases}\dfrac{d^2x}{d\tau^2}+K_s(\tau)x=-K_c(\tau)(\Delta\phi)x & (3.\ a)\\[2ex] \dfrac{d^2(\Delta\phi)}{d\tau^2}-2K_s(\tau)(\Delta\phi)=K_c(\tau)\left[(\Delta\phi)^2-\dfrac{(\kappa x)^2}{2}\right] & (3.\ b)\end{cases}$$

3. SMOOTHING

To obtain the smoothed equations of motion, we put

$$x=\chi\cdot(1+q_t)\ \text{and}\ \Delta\phi=\varphi\cdot(1+q_l). \tag{4}$$

q_t and q_l are periodic functions *i.e.* $q_t(\tau)=q_t(\tau+1)$ and $q_l(\tau)=q_l(\tau+1)$ with the conditions

$$\langle q_t\rangle=0=\langle\dot q_t\rangle\ \text{and}\ \langle q_l\rangle=0=\langle\dot q_l\rangle \tag{5}$$

where $^\bullet\equiv d/d\tau$ and the τ-dependent functions encolsed with < > indicate the averaged values over a focusing period. For example,

$$\langle q_t\rangle=\int_\tau^{\tau+1}q_t(\tau)d\tau.$$

Substitution of eqs.(4) into eq.(3.a) leads to

$$(1+q_t)\ddot\chi+(\langle K_s\rangle+q_tK_s)\chi+2\dot q_t\dot\chi=-K_c(1+q_t)(1+q_l)\chi\varphi. \tag{6}$$

Here, we have assumed that q_t satisfies the differential equation

$$\ddot q_t=-K_s+\langle K_s\rangle\ . \tag{7}$$

Averaging the both sides of eq.(6) and using the conditions (5), we have the smoothed transverse equation of motion as

$$\ddot\chi+\sigma_t^2\chi=-\sigma_a^2\chi\varphi \tag{8. a}$$

where

$$\sigma_t^2=\langle K_s\rangle+\langle q_tK_s\rangle\ \text{and}\ \sigma_a^2=\langle K_c(1+q_t)(1+q_l)\rangle\ .$$

Similarly, eq.(3.b) is smoothed to give

$$\ddot\varphi+\sigma_l^2\varphi=\sigma_b^2\varphi^2-\frac{1}{2}\kappa^2\sigma_c^2\chi^2 \tag{8, b}$$

where

$$\sigma_l^2=-2(\langle K_s\rangle+\langle q_lK_s\rangle),$$

$$\sigma_b^2=\langle K_c(1+q_l)^2\rangle \text{ and } \sigma_c^2=\langle K_c(1+q_t)^2\rangle \ .$$

And q_l satisfies

$$\ddot{q}_l=2(K_s-\langle K_s\rangle). \tag{9}$$

We can easily solve eqs.(7) and (9) under the conditions (5) and obtain

$$q_t=-\frac{1}{2}q_l=\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{C_n}{n^2}\sin(2n\pi\tau+\theta_n). \tag{10}$$

Making use of eq.(10), the phase advances are evaluated as follows;

$$\begin{cases}
\sigma_t^2=B+\dfrac{1}{8\pi^2}\displaystyle\sum_{n=1}^{\infty}(\frac{C_n}{n})^2 & (11.\ a)\\
\sigma_l^2=-2B+\dfrac{1}{2\pi^2}\displaystyle\sum_{n=1}^{\infty}(\frac{C_n}{n})^2 & (11.\ b)\\
\sigma_a^2=\left[1-\dfrac{1}{16\pi^4}\displaystyle\sum_{n=1}^{\infty}\left(\frac{C_n}{n^2}\right)^2\right]\cdot B'-\dfrac{1}{8\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{C_nC_n'}{n^2} & (11.\ c)\\
\sigma_b^2=\left[1+\dfrac{1}{8\pi^4}\displaystyle\sum_{n=1}^{\infty}\left(\frac{C_n}{n^2}\right)^2\right]\cdot B'-\dfrac{1}{2\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{C_nC_n'}{n^2} & (11.\ d)\\
\sigma_c^2=\left[1+\dfrac{1}{32\pi^4}\displaystyle\sum_{n=1}^{\infty}\left(\frac{C_n}{n^2}\right)^2\right]\cdot B'+\dfrac{1}{4\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{C_nC_n'}{n^2} & (11.\ e)
\end{cases}$$

4. SYNCHROBETATRON RESONANCES

Let us try to solve the coupled equations of motion eqs.(8) by using iteration method. We write the solutions as

$$\chi=\chi^{(1)}+\chi^{(2)}+\cdots \text{ and } \varphi=\varphi^{(1)}+\varphi^{(2)}+\cdots.$$

Substituting these expressions into eqs.(8), the first two lowest order equations of transverse and longitudinal motion are represented as follows;

$$\ddot{\chi}^{(1)}+\sigma_t^2\chi^{(1)}=0 \tag{12. a}$$

$$\ddot{\chi}^{(2)}+\sigma_t^2\chi^{(2)}=-\sigma_a^2\chi^{(1)}\varphi^{(1)} \tag{12. b}$$

$$\ddot{\varphi}^{(1)}+\sigma_l^2\varphi^{(1)}=0 \tag{13. a}$$

$$\ddot{\varphi}^{(2)}+\sigma_l^2\varphi^{(2)}=\sigma_b^2\varphi^{(1)^2}-\frac{1}{2}\kappa^2\sigma_c^2\chi^{(1)^2} \tag{13. b}$$

From eqs.(12.a) and (13.a), we have

$$\begin{cases}
\chi^{(1)}=\chi_1\cos(\sigma_t\tau+\mu_1) & (14.\ a)\\
\varphi^{(1)}=\varphi_1\cos(\sigma_l\tau+\mu_1'). & (14.\ b)
\end{cases}$$

Substitutions of eqs.(14) into eqs.(12.b) and (13.b) give the next order equations;

$$\begin{cases} \ddot{\chi}^{(2)}+\sigma_t^2\chi^{(2)}=-\dfrac{\sigma_a^2\chi_1\varphi_1}{2}\ \{\cos\,[(\sigma_t+\sigma_l)\tau+\mu_+]+\cos\,[(\sigma_t-\sigma_l)\tau+\mu_-]\} & (15.\ a) \\ \ddot{\varphi}^{(2)}+\sigma_l^2\varphi^{(2)}=\dfrac{\sigma_b^2\varphi_1^2}{2}\ [1+\cos(2\sigma_l\tau+2\mu_1')]-\dfrac{\kappa^2\sigma_c^2\chi_1^2}{4}\ [1+\cos(2\sigma_t\tau+2\mu_1)] & \\ & (15.\ b) \end{cases}$$

where $\mu_+=\mu_1+\mu_1'$ and $\mu_-=\mu_1-\mu_1'$.
The solutions of eqs.(15) which are zero and have zero derivatives at $\tau=0$ are

$$\chi^{(2)}=-\frac{\sigma_a^2}{2\sigma_t\sigma_l}\chi_1\varphi_1\ \Big\{2\ \sin(\sigma_t\tau+\mu_1)\sin(\frac{\sigma_l\tau}{2})\cos(\frac{\sigma_l\tau}{2}+\mu_1')$$
$$-\frac{\sigma_l}{2\sigma_t+\sigma_l}\sin\left[\frac{(2\sigma_t+\sigma_l)\tau}{2}\right]\sin(\frac{\sigma_l\tau}{2}+\mu_+)$$
$$+\frac{\sigma_l}{2\sigma_t-\sigma_l}\sin\left[\frac{(2\sigma_t-\sigma_l)\tau}{2}\right]\sin(\frac{\sigma_l\tau}{2}-\mu_-)\Big\} \qquad (16.\ a)$$

and

$$\varphi^{(2)}=\left[\varphi_1^2-\frac{1}{2}(\frac{\sigma_c}{\sigma_b}\kappa\chi_1)^2\right](\frac{\sigma_b}{\sigma_l}\sin\frac{\sigma_l\tau}{2})^2$$
$$+\frac{\sigma_b^2}{2\sigma_l^2}\varphi_1^2\ \Big\{\sin(\frac{\sigma_l\tau}{2})\sin(\frac{3\sigma_l\tau}{2}+2\mu_1')-\frac{1}{3}\sin(\frac{3\sigma_l\tau}{2})\sin(\frac{\sigma_l\tau}{2}+2\mu_1')\Big\}$$
$$+\frac{\sigma_c^2}{2\sigma_l^2}\kappa^2\chi_1^2\ \Big\{\frac{\sigma_l}{2\sigma_t+\sigma_l}\sin\left[\frac{(2\sigma_t+\sigma_l)\tau}{2}\right]\sin\left[\frac{(2\sigma_t-\sigma_l)\tau}{2}+2\mu_1\right]$$

The coupling terms in eqs.(8) yield synchrobetatron resonances. As is shown in eqs.(16), a resonance condition is written

$$2\sigma_t-\sigma_l=0. \qquad (17)$$

We see from eqs.(2) that the synchrobetatron coupling terms have the coordinate dependences like $(\Delta\phi)^n x^{2m}$ (n,m=integer) which result in the various resonance conditions given generally by

$$n_t\cdot\sigma_t\pm n_l\cdot\sigma_l=m$$

where n_t, n_l and m are integers. However, eq.(17) corresponds to the lowest order resonance which is the most severe one. Furthermore, as is mentioned in the next section, this resonance have the largest unstable region around the condition (17) compared with the other resonances. Using eqs.(11.a) and (11.b), eqs.(17) is shown to be equivalent to B=0. This means that an APF linac having symmetric phase sequences must be avoided to eliminate the lowest order resonance.

5. UNSTABLE REGION

In this section, we estimate the width of the unstable region around the resonance given by the condition (17). Making use of the expression of $\varphi^{(1)}$ instead of φ in r.h.s. of eq.(8.a), we have

$$\ddot{\chi}+\sigma_t^2\left[1+\left(\frac{\sigma_a}{\sigma_t}\right)^2\varphi_1\cos(\sigma_l\tau+\mu_1')\right]\chi=0. \tag{18}$$

To calculate the unstable width around the $2\sigma_t=\sigma_1$ resonance, we put $\sigma_1=2\sigma_t+\delta$ to rewrite eq.(18) as

$$\ddot{\chi}+\sigma_t^2\ [1+\eta_1\cos(2\sigma_t+\delta)\tau]\ \chi=0 \tag{19}$$

where $\eta_1=(\sigma_a/\sigma_t)^2\varphi_1$.

Now, we try to find an approximated solution having the form

$$\chi(\tau)=a_c(\tau)\cos\left(\sigma_t+\frac{\delta}{2}\right)\tau+a_s(\tau)\sin\left(\sigma_s+\frac{\delta}{2}\right)\tau \tag{20}$$

where $a_c(\tau)$ and $a_s(\tau)$ are slowly varying functions whose first τ-derivatives are the same order as δ. Substitution of eq.(20) into eq.(19) with the neglection of the second derivative terms leads to

$$\begin{cases} pa_c+\dfrac{1}{2}\left(\delta+\dfrac{\eta\sigma_t}{2}\right)a_s=0 & (21.\ a)\\[2ex] \dfrac{1}{2}\left(\delta-\dfrac{\eta\sigma_t}{2}\right)a_c-pa_s=0\,. & (21.\ b)\end{cases}$$

Here, we assumed the exp($p\tau$)-dependence for $a_c(\tau)$ and $a_s(\tau)$. The condition that the set of equations (21) have non-trivial solutions are given by

$$p^2=\frac{1}{4}\left[\left(\frac{\eta\sigma_t}{2}\right)^2-\delta^2\right]\ . \tag{22}$$

Since the particle motion becomes unstable if $p^2>0$, the unstable region is evaluated from eq.(22) as

$$1-\frac{1}{4}\left(\frac{\sigma_a}{\sigma_t}\right)^2|\varphi_1|<\frac{\sigma_l}{2\sigma_t}<1+\frac{1}{4}\left(\frac{\sigma_a}{\sigma_t}\right)^2|\varphi_1|\,. \tag{23}$$

This is the unstable width corresponding to the lowest order parametric resonance derived from the approximated solution (20) with the assumption $\sigma_1=2\sigma_t+\delta$. This kind of resonances also occurs when $\sigma_1=2\sigma_t/n$, but the unstable width of the n-th order resonance is proportional to η^n and the beam growth rate becomes smaller with increasing resonance order. Furthermore, in an usual APF linac, we generally set $\sigma_1 \gtrsim 2\sigma_t$ to make the longitudinal acceptance as large as possible, so the resonance (17) is the most dangerous one.

6. CONCLUDING REMARKS

Coupled motion in APF linacs has been described, starting with the smoothed equations of motion based on the same type equations presented in Ref. 3. The equations of motion analyzed here are obtained by using the stepping field approximation method, generalizing the usual equations for normal DTLs. In fact, if we put $N=1$ and $C_n=0=C_n'$, we can easily show that all equations in this paper agree with those describing the synchrobetatron motion in normal DTLs. (However, in this case, it is necessary to add the contribution from external focusing elements to the transverse phase advance σ_t.)

It has been shown from the smoothed equations that particle motion in a symmetric APF (SAPF) linac may be affected by the lowest order coupling effect because the longitudinal phase advance in SAPF is twice as large as the transverse one. In usual designs of AAPF and MAPF, we generally choose phase sequences so that $\sigma_l \gtrsim 2\sigma_t$, to make the longitudinal acceptance as large as possible. Therefore, in practical meanings, the lowest order synchrobetatron resonance imposes a severe restriction to APF phase sequence designs and, from eq. (23), the relation

$$\sigma_l > 2\sigma_t \cdot \left[1+\left(\frac{\sigma_a}{2\sigma_t}\right)^2 |\varphi_l|\right]$$

must be satisfied to avoid the resonance with the simultaneous achievement of the large longitudinal acceptance. In this formula, φ_l is approximately equal to the half-width of the longitudinal phase acceptable region evaluated graphically from the effective potential presented in Ref. 3.

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APPENDIX A

NOTATIONS

β, γ = relativistic parameters
c = speed of light
m_0 = particle mass
e = electron charge
λ = rf wavelength
q = charge state of ion
$\phi^s, \phi_k{}^s$ = equilibrium phase
V, V_k = intergap voltage amplitude
T, T_k = transit-time factor
N = number of gaps in a focusing period
L = cell length
L_N = total length of a focusing period
g, g_k = gap width
x = transverse coordinate of particle
$\Delta\phi$ = phase difference of non-equilibrium and equilibrium particle

All the above symbols with subscript k represent the values of the k-th cell in a focusing period.

APPENDIX B

SINGLE PARTICLE EQUATIONS OF MOTION IN APF LINACS

First of all, we try to obtain the electromagnetic field in a drift tube structure whose periodic length is L_N. Here, we consider an even-π mode DTL with axisymmetric intergap field. The notations of the symbols in this paper are listed in APPENDIX A. Using the cylindrical coordinate system, the Maxwell's equations are represented as follows;

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial(rE_r)}{\partial r}+\dfrac{\partial E_z}{\partial z}=0 \\ \dfrac{\partial E_r}{\partial z}-\dfrac{\partial E_z}{\partial r}=-\dfrac{\partial B_\theta}{\partial t} \\ -\dfrac{\partial B_\theta}{\partial z}=\dfrac{1}{c^2}\dfrac{\partial E_r}{\partial t} \\ \dfrac{1}{r}\dfrac{\partial(rB_\theta)}{\partial r}=\dfrac{1}{c^2}\dfrac{\partial E_z}{\partial t} \\ E_\theta=B_r=B_z=0 \end{cases}$$

Then, the axial electric field E_z satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right)+\frac{\partial^2 E_z}{\partial z^2}=-\left(\frac{\omega}{c}\right)^2 E_z \tag{B. 1}$$

with a standing wave expression

$$E_z(z,r,t)=E_z(z,r)\cos\,\omega t.$$

Since E_z can be expanded as a Fourier series

$$E_z(z,r)=\sum_{n=0}^{\infty}\varepsilon_n(r)\cos\left(\frac{2n\pi z}{L_N}\right),$$

eq. (B.1) reduces to

$$\frac{\partial^2\varepsilon_n}{\partial r^2}+\frac{1}{r}\frac{\partial\varepsilon_n}{\partial \mathrm{r}}-\mathrm{k}_n{}^2\varepsilon_n=0$$

which has a solution $\varepsilon_n(\mathrm{r})=\mathrm{A}_n\mathrm{I}_0(\mathrm{k}_n\mathrm{r})$ where $\mathrm{I}_n(\mathrm{x})$ indicates the modified Bessel function of n-th order and $\mathrm{k}_n{}^2=(2\pi/\lambda)^2\,[(\mathrm{n}\lambda/\mathrm{L}_N)^2-1]$. Accordingly, we get the electromagnetic fields

$$\begin{cases} E_z(z,r,t)=\sum_{n=0}^{\infty} A_n I_0(k_n r)\cos(\frac{2n\pi z}{L_N})\cos\,\omega t & (B.2a) \\ E_r(z,r,t)=\sum_{n=0}^{\infty}\frac{2n\pi A_n}{k_n L_N} I_1(k_n r)\sin(\frac{2n\pi z}{L_N})\cos\,\omega t & (B.2b) \\ B_\theta(z,r,t)=-\sum_{n=0}^{\infty}\frac{A_n\omega}{k_n c^2} I_1(k_n r)\cos(\frac{2n\pi z}{L_N})\sin\,\omega t & (B.2c) \end{cases}$$

by using the Maxwell's equatins. If we use the averaged Lorentz force over single structure period, the transverse equation of motion is given by

$$\frac{dp_x}{d\tau}=-\frac{\mathrm{eq}\,VT}{c\beta\gamma}\mathrm{I}_1(\kappa x)\sin\phi \qquad (B.3)$$

from eqs.(B.2) under the assumption that $L_N=N\beta\lambda$ and $\omega t=(2\pi/\beta\lambda)z+\phi$. In eq.(B.3), we put $A_N=2VT/L_N$, $\kappa=2\pi/\beta\gamma\lambda$ and $\tau=\beta ct/L_N$. Now, putting $\phi=\Delta\phi+\phi^s$ and applying the stepping field approximation method proposed in Ref. 3 to eq.(B.3), we obtain the transverse equation of motion in an APF structure

$$\frac{dp_x}{d\tau}=-\frac{\mathrm{eq}}{c\beta\gamma} I_1(\kappa x)\sum_{k=1}^{N} V_k T_k \Lambda_k(\tau)\sin(\Delta\phi+\phi_k{}^s) \qquad (B.4a)$$

where

$$\Lambda_k(\tau)=1+2\sum_{n=1}^{\infty} S_{nk}\cos\,[2n\pi(\tau-\tau_k)]\ ,\quad S_{nk}=\frac{\sin(n\pi g_k/L_N)}{n\pi g_k/L_N},$$

and

$$\tau_k(k\neq 1)=\frac{\beta\lambda}{2\pi L_N}\sum_{m=1}^{k-1}(\phi_{m+1}{}^s-\phi_m{}^s+\eta\pi)\ \mathit{with}\ \tau_1=0.$$

η is an accelerating mode number and, for example, $\eta=2$ for a 2π- mode DTL. The momentum p_x is related to the transverse coordinate x as

$$\frac{dx}{d\tau}=\frac{L_N}{m_0 c\beta\gamma}\cdot p_x. \qquad (B.\ 4b)$$

If we assume that β is approximately constant through single focusing period, eqs. (B. 4) give

$$\frac{d^2x}{d\tau^2}+\frac{eqL_N}{m_0c^2\beta^2\gamma^2} I_1(\kappa x)\sum_{k=1}^{N} V_k T_k \Lambda_k(\tau)\sin(\Delta\phi+\phi_k{}^s)=0.$$

From eq. (B. 2a), we see that the energy difference between non-equilibrium and equilibrium particle satisfies the equation

$$\frac{d(\Delta W)}{d\tau}=eqVT\ [I_0(\kappa x)\cos\phi-\cos\phi^s]\ , \qquad (B.\ 5)$$

assuming that the change of radial particle position in a gap is small enough to be neglected. Again, we apply the stepping field method to eq.(B.5) to obtain

$$\frac{d(\Delta W)}{d\tau}=eq\sum_{k=1}^{N}V_kT_k\Lambda_k(\tau)\ \left[I_0(\varkappa x)\cos(\Delta\phi+\phi_k{}^s)-\cos\phi_k{}^s\right]\ . \tag{B. 6a}$$

The τ-derivative of $\Delta\phi$ approximately satisfies

$$\frac{d(\Delta\phi)}{d\tau}=-\frac{2\pi L_N}{\lambda}\cdot\frac{\Delta \mathrm{W}}{\mathrm{m_0c^2}\beta^3\gamma^3.} \tag{B. 6b}$$

Then, we have from eqs.(B.6) the longitudinal equation of motion of an APF structure

$$\frac{d^2(\Delta\phi)}{d\tau^2}+\frac{2\pi eqL_N}{m_0c^2\beta^3\gamma^3\lambda}\sum_{k=1}^{N}V_kT_k\Lambda_k(\tau)\ \left[I_0(\varkappa x)\cos(\Delta\phi+\phi_k{}^s)-\cos\phi_k{}^s\right]=0.$$

We can easily obtain the same equations of motion also in an odd-π mode structure, following the similar procedure described above.